Test 2, Math 111 **PLEDGED** Key

Show all work. Write legibly and lucidly. (110 points possible)

1. Determine (and explain) whether or not the Mean Value Theorem is applicable to the given function over the given interval. If the theorem is applicable, find the number c guaranteed by the theorem. *(18 points total. Functions where the mean value theorem applies will be more heavily weighted.)*

a) $f(x) = x^2 - 4x$ over $[2, 6]$

1.) $f(x) = x^2 - 4x$ is continuous over $[2,6]$ since f is a polynomial

2.) $f'(x) = 2x - 4$ is continuous over $(2,6)$ since f' is a line

Therefore, the Mean Value Theorem applies

Slope of secant line: $f(2) = 4 - 8 = -4$ $(2,-4)$

$f(6) = 36 - 24 = 12$ $(6,12)$

$$m = \frac{12+4}{6-2} = \frac{16}{4} = 4$$

$f'(c) = 2c - 4 = 4$, $2c = 8$, $c = 4$ and

$a = 2 < c = 4 < b = 6$

b) $f(x) = x^{1/3} - 1$ over $[-1, 1]$

$f(x) = x^{1/3} - 1$ is continuous over $[-1,1]$; however

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3x^{2/3}}$ is undefined a $x = 0$ so

$f(x)$ is not differentiable over $(-1,1)$.

So, Mean Value Theorem does not apply.

2. Air is being pumped into a spherical balloon at the rate of 8π in^3/min. Find the rate of change of the radius when the surface area is 16π in^2. *(8 points)*

$$V = \frac{4}{3}\pi r^3 \qquad SA = 4\pi r^2$$

$$\frac{dV}{dt} = 8\pi \text{ in}^3/\text{min} \qquad \frac{dr}{dt} = ? \qquad \underline{\text{when:}}\ SA = 16\pi \text{ in}^2$$

$$4\pi r^2 = 16\pi$$

$$r^2 = 4$$

$$r = \pm 2, \text{ so}$$

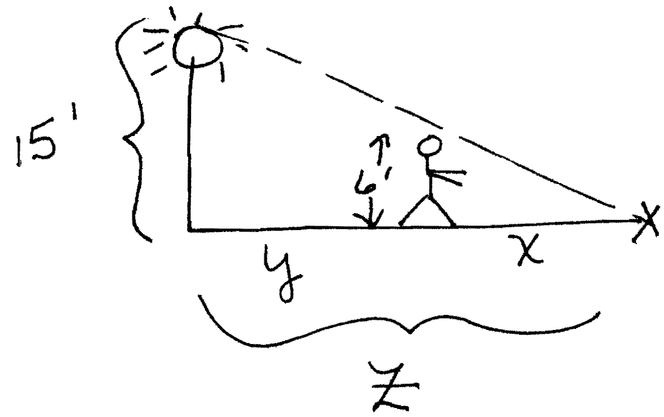
$$r = 2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{8\cancel{\pi}}{4\cancel{\pi}(2)^2} = \frac{dr}{dt}$$

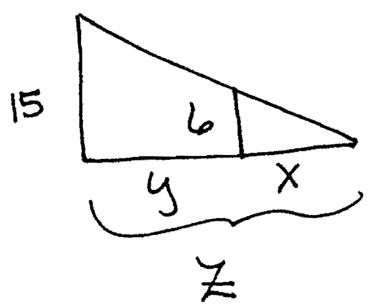
$$\boxed{\frac{dr}{dt} = \frac{1}{2} \text{ in/min}}$$

3. A man 6 feet tall walks away from a street light 15 feet high at a rate of 6 ft/sec. How fast is the far end of his shadow moving when he is 30 feet from the light pole? *(10 points)*



$$\frac{dz}{dt} = ? \text{ when } y = 30$$

$$\frac{dy}{dt} = 6\,\text{ft/sec}$$



$$z = x + y$$

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{dz}{dt} = 4\,\text{ft/sec} + 6\,\text{ft/sec} = \boxed{10\,\text{ft/sec}}$$

Similar Δs

$$\frac{15}{z} = \frac{6}{x}$$

$$15x = 6z = 6(x+y) = 6x + 6y$$

$$15\frac{dx}{dt} = 6\frac{dx}{dt} + 6\frac{dy}{dt}$$

$$9\frac{dx}{dt} = 6\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{2}{3}\frac{dy}{dt} = \frac{2}{\cancel{3}}\cdot\overset{2}{\cancel{6}}\,\text{ft/sec} = \underline{\underline{4\,\text{ft/sec}}}$$

4. For each of the functions given, determine and fill in the information requested (give both coordinates for intercepts and other points), and sketch the graph of the function on the given axes, labeling all important points and features.

Show all work on separate paper.

a) $f(x) = 3x^5 - 5x^3$ (16 points)

intercept $0 = x^3(3x^2-5) \rightarrow x=0,\ x=\pm\sqrt{\frac{5}{3}}$

$f'(x) = 15x^4 - 15x^2$

$15x^2(x^2-1)$

$x=0 \quad x=\pm 1$

$+ \quad - \quad - \quad +$

$-1 \quad 0 \quad +1$

MAX MIN

Intercept(s): $(0,0)\ (\sqrt{\frac{5}{3}},0)\ (-\sqrt{\frac{5}{3}},0)$

Interval(s) of increase: $x<-1 \quad x>1$

Point(s) where the tangent line is horizontal: $(0,0)\ (-1,2)\ (1,-2)$

Relative (i.e., local) minima: $(1,-2)$

Relative (i.e., local) maxima: $(-1,2)$

Interval(s) where the graph is concave upward: $-\sqrt{\frac{1}{2}}<x<0,\ x>\sqrt{\frac{1}{2}}$

Points of Inflection: $(0,0)\ (-\sqrt{\frac{1}{2}},\frac{7}{4\sqrt{2}})\ (\sqrt{\frac{1}{2}},\frac{-7}{4\sqrt{2}})$

$f''(x) = 60x^3 - 30x$

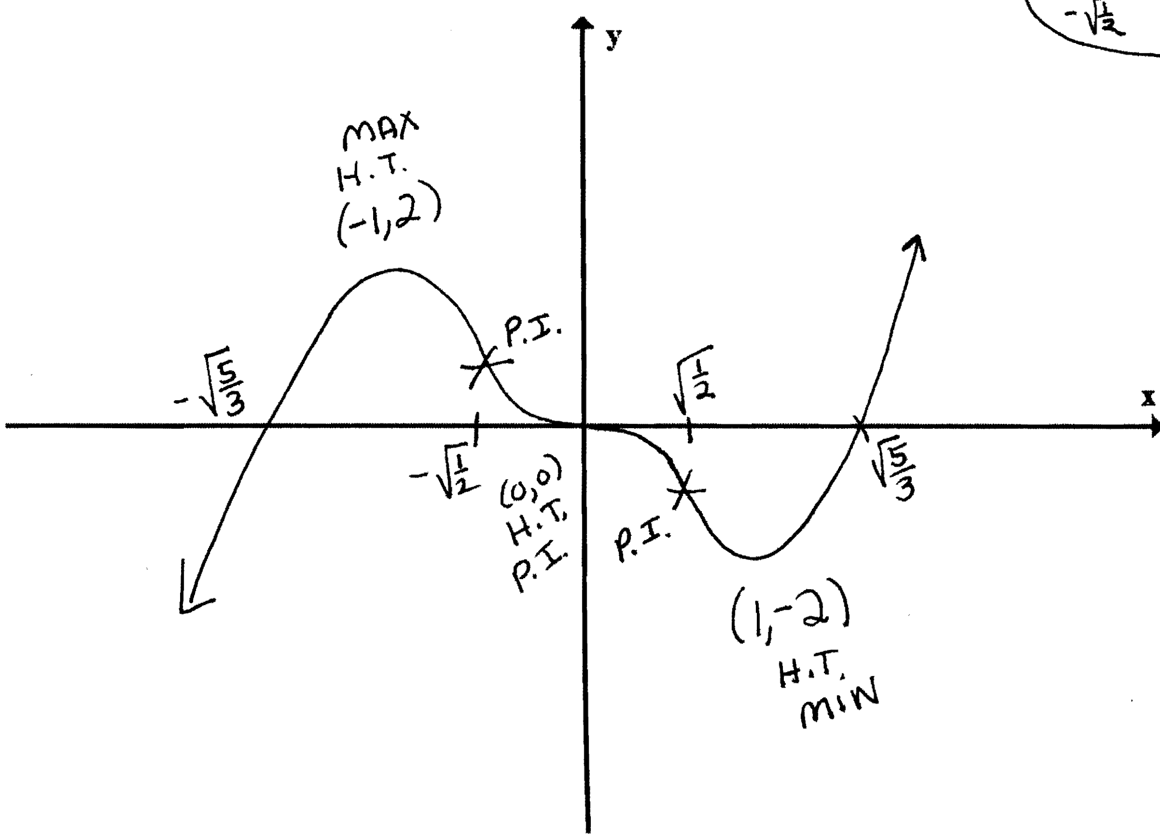
$30x(2x^2-1)$

$x=0$

$x=\pm\sqrt{\frac{1}{2}}$

$- \quad + \quad - \quad +$

$-\sqrt{\frac{1}{2}} \quad 0 \quad \sqrt{\frac{1}{2}}$



b)

$$f(x) = \frac{x^3}{(x-2)^2}$$ *(20 points)*

Intercept(s): $(0,0)$

Asymptote(s): $x=2$

Interval(s) of increase: $x<0,\ 0<x<2,\ x>6$

Point(s) where the tangent line is horizontal: $(6, 27/2)\ (0,0)$

Relative (i.e., local) minima: $(6, 27/2)$

Relative (i.e., local) maxima: NONE

Interval(s) where concave upward: $0<x<2 \quad x>2$

Points of Inflection: $(0,0)$

$$f'(x) = \frac{(x-2)^2 \cdot 3x^2 - x^3 \cdot 2(x-2)}{(x-2)^4}$$

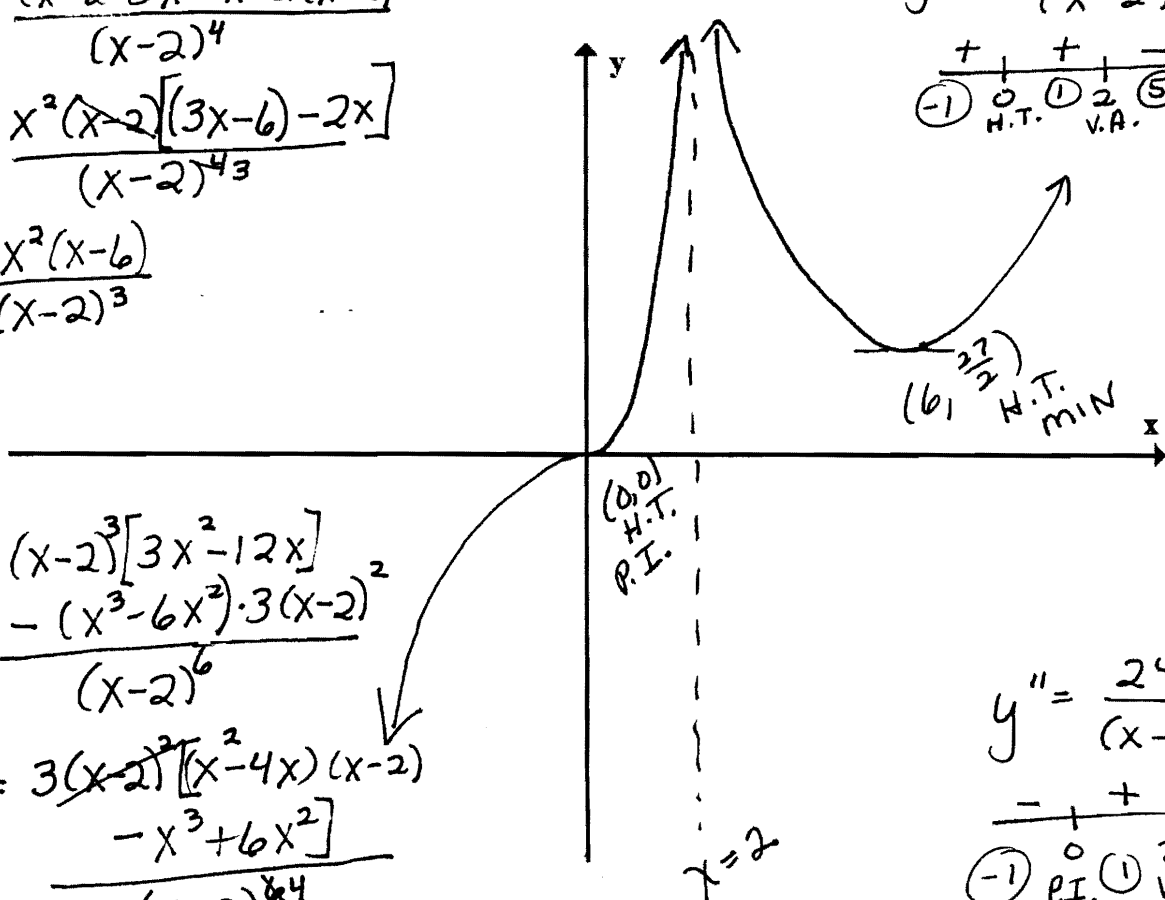
$$f'(x) = \frac{x^2\cancel{(x-2)}[(3x-6)-2x]}{(x-2)^{\cancel{4}3}}$$

$$f'(x) = \frac{x^2(x-6)}{(x-2)^3}$$

$$y' = \frac{x^2(x-6)}{(x-2)^2}$$

$+$ | $+$ | $-$ | $+$

-1 0 H.T. 1 2 V.A. 5 6 H.T. MIN 10



$$f''(x) = \frac{(x-2)^3[3x^2-12x] - (x^3-6x^2)\cdot 3(x-2)^2}{(x-2)^6}$$

$$f''(x) = \frac{3\cancel{(x-2)^2}[(x^2-4x)(x-2) - x^3+6x^2]}{(x-2)^{\cancel{6}4}}$$

$$= \frac{3(\cancel{x^3}-\cancel{4x^2}-\cancel{2x^2}+8x - \cancel{x^3}+\cancel{6x^2})}{(x-2)^4} = \frac{24x}{(x-2)^4}$$

$$y'' = \frac{24x}{(x-2)^4}$$

$-$ | $+$ | $+$

-1 0 P.I. 1 2 V.A. 3

c) $y = x^{1/3}(x+1)^{2/3}$ **(omit concavity considerations)** *(20 points)*

Intercept(s): $(0,0)\ (-1,0)$

Asymptote(s): NONE

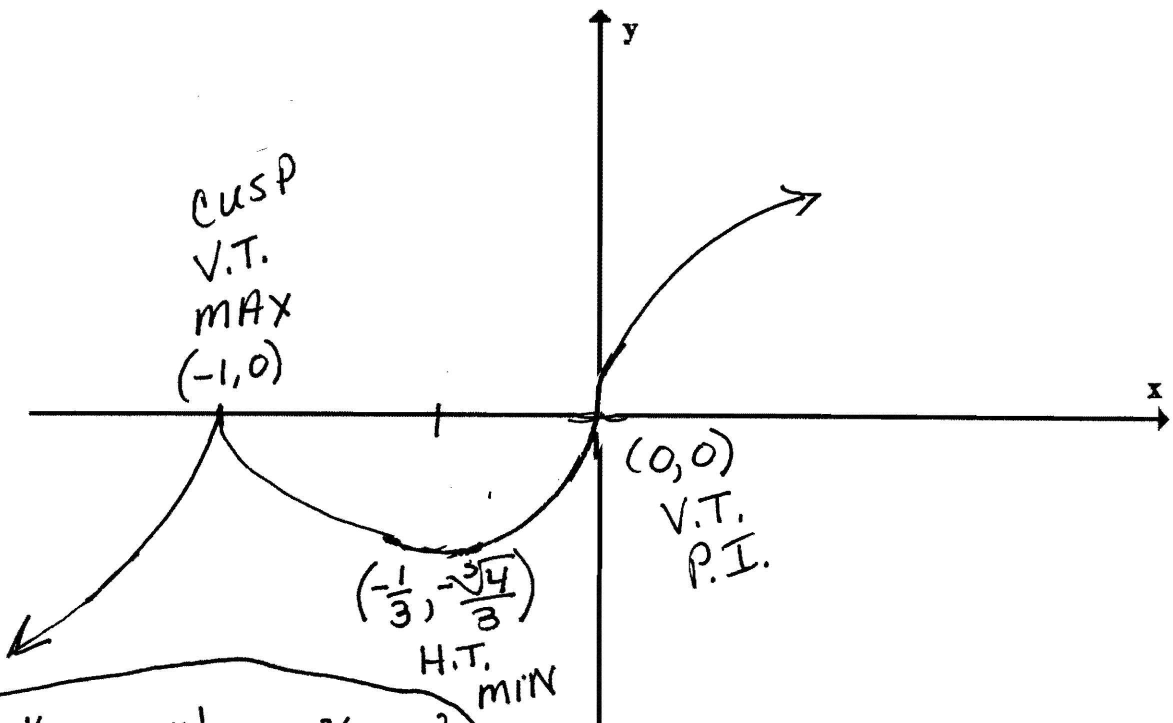
Interval(s) of decrease: $-1 < x < -\frac{1}{3}$

Point(s) where the tangent line is horizontal: $\left(-\frac{1}{3}, -\frac{\sqrt[3]{4}}{3}\right)$

Point(s) where there is a cusp or vertical tangent: $(-1,0)\ (0,0)$

Relative (i.e., local) minima: $\left(-\frac{1}{3}, -\frac{\sqrt[3]{4}}{3}\right)$

Relative (i.e., local) maxima: $(-1,0)$



$$y' = x^{1/3}\cdot\frac{2}{3}(x+1)^{-\frac{1}{3}} + (x+1)^{2/3}\cdot\frac{1}{3}x^{-\frac{2}{3}}$$

$$= \frac{2x^{1/3}}{3(x+1)^{1/3}} + \frac{(x+1)^{2/3}}{3x^{2/3}} = \frac{2x+x+1}{3x^{2/3}(x+1)^{1/3}}$$

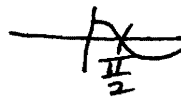
$$y' = \frac{3x+1}{3x^{2/3}(x+1)^{1/3}}$$

$$\begin{array}{ccccccc} + & & - & & + & & + \\ & -1 & & -\frac{1}{3} & & 0 & \end{array}$$

5. Evaluate all limits shown. Show all necessary work.
(3 points each. 18 points total.)

a) $\lim\limits_{x\to\pi/2+} \tan x$ $\qquad$ $-\infty$

$$\lim_{x\to\frac{\pi}{2}+} \frac{\sin x}{\cos x} = -\infty$$



$y = \cos x$ is neg on rt

b) $\lim\limits_{x\to\infty} \dfrac{e^x}{2e^x - 1}$ $\qquad$ $\frac{1}{2}$

$$\lim_{x\to\infty} \frac{e^x}{2e^x-1}\cdot\frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty}\frac{1}{2-\cancel{\frac{1}{e^x}}\to 0}$$

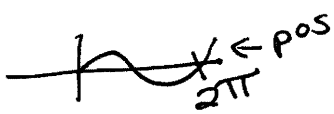
c) $\lim\limits_{x\to-\infty} \dfrac{3-2x^3}{x^2+1}\cdot\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$ $\qquad$ $+\infty$

$$\lim_{x\to-\infty}\frac{\cancel{\frac{3}{x^2}}\to 0 - 2x}{1+\cancel{\frac{1}{x^2}}\to 0} = +\infty$$

d) $\lim\limits_{x\to\infty} \dfrac{x^{2/3}}{x+1} = 0$ $\qquad$ 0

e) $\lim\limits_{x\to2\pi+} \csc x$ $\qquad$ $+\infty$

$$\lim_{x\to 2\pi+}\frac{1}{\sin x}$$



f) $\lim\limits_{x\to-2} \dfrac{x^3+8}{x+2}$ $\qquad$ 12

$$\lim_{x\to-2}\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}} = 4+4+4$$